

## THREE DIMENSIONAL GEOMETRY

### 11.1 Overview

**11.1.1** Direction cosines of a line are the cosines of the angles made by the line with positive directions of the co-ordinate axes.

**11.1.2** If  $l, m, n$  are the direction cosines of a line, then  $l^2 + m^2 + n^2 = 1$

**11.1.3** Direction cosines of a line joining two points P ( $x_1, y_1, z_1$ ) and Q ( $x_2, y_2, z_2$ ) are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ},$$

where  $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**11.1.4** Direction ratios of a line are the numbers which are proportional to the direction cosines of the line.

**11.1.5** If  $l, m, n$  are the direction cosines and  $a, b, c$  are the direction ratios of a line,

$$\text{then } l = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}; n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

**11.1.6** Skew lines are lines in the space which are neither parallel nor intersecting. They lie in the different planes.

**11.1.7** Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

**11.1.8** If  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are the direction cosines of two lines and  $\theta$  is the acute angle between the two lines, then

$$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

**11.1.9** If  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are the directions ratios of two lines and  $\theta$  is the acute angle between the two lines, then

$$\cos \theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}} \right|$$

**11.1.10** Vector equation of a line that passes through the given point whose position vector is  $\vec{a}$  and parallel to a given vector  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda \vec{b}$ .

**11.1.11** Equation of a line through a point  $(x_1, y_1, z_1)$  and having directions cosines  $l, m, n$  (or, direction ratios  $a, b$  and  $c$ ) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{or} \quad \left( \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right).$$

**11.1.12** The vector equation of a line that passes through two points whose position vectors are  $\vec{a}$  and  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ .

**11.1.13** Cartesian equation of a line that passes through two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

**11.1.14** If  $\theta$  is the acute angle between the lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ , then

$$\theta \text{ is given by } \cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|} \quad \text{or} \quad \theta = \cos^{-1} \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}.$$

**11.1.15** If  $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$  and  $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$  are equations of two lines, then the acute angle  $\theta$  between the two lines is given by  $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$ .

**11.1.16** The shortest distance between two skew lines is the length of the line segment perpendicular to both the lines.

**11.1.17** The shortest distance between the lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$  is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|.$$

**11.1.18** Shortest distance between the lines:  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and

$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}$$

**11.1.19** Distance between parallel lines  $\vec{r} = \vec{a}_1 + \mu \vec{b}$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}$  is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|.$$

**11.1.20** The vector equation of a plane which is at a distance  $p$  from the origin, where  $\hat{n}$  is the unit vector normal to the plane, is  $\vec{r} \cdot \hat{n} = p$ .

**11.1.21** Equation of a plane which is at a distance  $p$  from the origin with direction cosines of the normal to the plane as  $l, m, n$  is  $lx + my + nz = p$ .

**11.1.22** The equation of a plane through a point whose position vector is  $\vec{a}$  and perpendicular to the vector  $\vec{n}$  is  $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$  or  $\vec{r} \cdot \vec{n} = d$ , where  $d = \vec{a} \cdot \vec{n}$ .

**11.1.23** Equation of a plane perpendicular to a given line with direction ratios  $a, b, c$  and passing through a given point  $(x_1, y_1, z_1)$  is  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ .

**11.1.24** Equation of a plane passing through three non-collinear points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

**11.1.25** Vector equation of a plane that contains three non-collinear points having position vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  is  $(\vec{r}-\vec{a}) \cdot [(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})] = 0$

**11.1.26** Equation of a plane that cuts the co-ordinates axes at  $(a, 0, 0)$ ,  $(0, b, 0)$  and  $(0, 0, c)$  is  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .

**11.1.27** Vector equation of any plane that passes through the intersection of planes  $\vec{r} \cdot \vec{n}_1 = d_1$  and  $\vec{r} \cdot \vec{n}_2 = d_2$  is  $(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$ , where  $\lambda$  is any non-zero constant.

**11.1.28** Cartesian equation of any plane that passes through the intersection of two given planes  $A_1x + B_1y + C_1z + D_1 = 0$  and  $A_2x + B_2y + C_2z + D_2 = 0$  is  $(A_1x + B_1y + C_1z + D_1) + \lambda (A_2x + B_2y + C_2z + D_2) = 0$ .

**11.1.29** Two lines  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$  are coplanar if  $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$

**11.1.30** Two lines  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$$

**11.1.31** In vector form, if  $\theta$  is the acute angle between the two planes,  $\vec{r} \cdot \vec{n}_1 = d_1$  and

$$\vec{r} \cdot \vec{n}_2 = d_2, \text{ then } \theta = \cos^{-1} \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

**11.1.32** The acute angle  $\theta$  between the line  $\vec{r} = \vec{a} + \lambda \vec{b}$  and plane  $\vec{r} \cdot \vec{n} = d$  is given by

$$\sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| \cdot |\vec{n}|}$$

## 11.2 Solved Examples

### Short Answer (S.A.)

**Example 1** If the direction ratios of a line are 1, 1, 2, find the direction cosines of the line.

**Solution** The direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Here  $a, b, c$  are 1, 1, 2, respectively.

$$\text{Therefore, } l = \frac{1}{\sqrt{1^2 + 1^2 + 2^2}}, m = \frac{1}{\sqrt{1^2 + 1^2 + 2^2}}, n = \frac{2}{\sqrt{1^2 + 1^2 + 2^2}}$$

i.e.,  $l = \frac{1}{\sqrt{6}}, m = \frac{1}{\sqrt{6}}, n = \frac{2}{\sqrt{6}}$  i.e.  $\pm \left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)$  are D.C.'s of the line.

**Example 2** Find the direction cosines of the line passing through the points P (2, 3, 5) and Q (-1, 2, 4).

**Solution** The direction cosines of a line passing through the points P ( $x_1, y_1, z_1$ ) and Q ( $x_2, y_2, z_2$ ) are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

Here  $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{(-1 - 2)^2 + (2 - 3)^2 + (4 - 5)^2} = \sqrt{9 + 1 + 1} = \sqrt{11}$$

Hence D.C.'s are

$$\pm \left( \frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}} \right) \text{ or } \pm \left( \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right).$$

**Example 3** If a line makes an angle of  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  with the positive direction of  $x$ ,  $y$ ,  $z$ -axes, respectively, then find its direction cosines.

**Solution** The direction cosines of a line which makes an angle of  $\alpha$ ,  $\beta$ ,  $\gamma$  with the axes, are  $\cos\alpha$ ,  $\cos\beta$ ,  $\cos\gamma$

Therefore, D.C.'s of the line are  $\cos 30^\circ$ ,  $\cos 60^\circ$ ,  $\cos 90^\circ$  i.e.,  $\pm \left( \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right)$

**Example 4** The  $x$ -coordinate of a point on the line joining the points Q (2, 2, 1) and R (5, 1, -2) is 4. Find its  $z$ -coordinate.

**Solution** Let the point P divide QR in the ratio  $\lambda : 1$ , then the co-ordinate of P are

$$\left( \frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1} \right)$$

But  $x$ -coordinate of P is 4. Therefore,

$$\frac{5\lambda+2}{\lambda+1} = 4 \Rightarrow \lambda = 2$$

Hence, the  $z$ -coordinate of P is  $\frac{-2\lambda+1}{\lambda+1} = -1$ .

**Example 5** Find the distance of the point whose position vector is  $(2\hat{i} + \hat{j} - \hat{k})$  from the plane  $\vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) = 9$

**Solution** Here  $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ ,  $\vec{n} = \hat{i} - 2\hat{j} + 4\hat{k}$  and  $d = 9$

So, the required distance is  $\frac{|(2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) - 9|}{\sqrt{1+4+16}}$

$$= \frac{|2-2-4-9|}{\sqrt{21}} = \frac{13}{\sqrt{21}}$$

**Example 6** Find the distance of the point  $(-2, 4, -5)$  from the line  $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

**Solution** Here  $P(-2, 4, -5)$  is the given point.

Any point  $Q$  on the line is given by  $(3\lambda - 3, 5\lambda + 4, (6\lambda - 8))$ ,

$$\overline{PQ} = (3\lambda - 1)\hat{i} + 5\lambda\hat{j} + (6\lambda - 3)\hat{k}$$

Since  $\overline{PQ} \perp (3\hat{i} + 5\hat{j} + 6\hat{k})$ , we have

$$3(3\lambda - 1) + 5(5\lambda) + 6(6\lambda - 3) = 0$$

$$9\lambda + 25\lambda + 36\lambda = 21, \text{ i.e. } \lambda = \frac{3}{10}$$

Thus  $\overline{PQ} = -\frac{1}{10}\hat{i} + \frac{15}{10}\hat{j} - \frac{12}{10}\hat{k}$

Hence  $|\overline{PQ}| = \frac{1}{10}\sqrt{1+225+144} = \sqrt{\frac{37}{10}}$

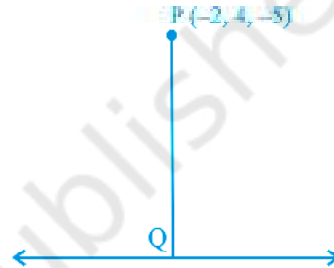


Fig. 11.1

**Example 7** Find the coordinates of the point where the line through  $(3, -4, -5)$  and  $(2, -3, 1)$  crosses the plane passing through three points  $(2, 2, 1)$ ,  $(3, 0, 1)$  and  $(4, -1, 0)$

**Solution** Equation of plane through three points  $(2, 2, 1)$ ,  $(3, 0, 1)$  and  $(4, -1, 0)$  is

$$[(\vec{r} - (2\hat{i} + 2\hat{j} + \hat{k})) \cdot ((\hat{i} - 2\hat{j}) \times (\hat{i} - \hat{j} - \hat{k}))] = 0$$

i.e.  $\vec{r} \cdot (2\hat{i} + \hat{j} + \hat{k}) = 7$  or  $2x + y + z - 7 = 0 \dots (1)$

Equation of line through  $(3, -4, -5)$  and  $(2, -3, 1)$  is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \dots (2)$$

Any point on line (2) is  $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$ . This point lies on plane (1). Therefore,  $2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$ , i.e.,  $\lambda = z$

Hence the required point is  $(1, -2, 7)$ .

**Long Answer (L.A.)**

**Example 8** Find the distance of the point  $(-1, -5, -10)$  from the point of intersection of the line  $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$  and the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ .

**Solution** We have  $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$  and  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$

Solving these two equations, we get  $[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})] \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$  which gives  $\lambda = 0$ .

Therefore, the point of intersection of line and the plane is  $(2, -1, 2)$  and the other given point is  $(-1, -5, -10)$ . Hence the distance between these two points is

$$\sqrt{[2 - (-1)]^2 + [-1 - (-5)]^2 + [2 - (-10)]^2}, \text{ i.e. } 13$$

**Example 9** A plane meets the co-ordinates axis in A, B, C such that the centroid of the  $\Delta ABC$  is the point  $(\alpha, \beta, \gamma)$ . Show that the equation of the plane is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

**Solution** Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Then the co-ordinate of A, B, C are  $(a, 0, 0)$ ,  $(0, b, 0)$  and  $(0, 0, c)$  respectively. Centroid of the  $\Delta ABC$  is

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \text{ i.e. } \left( \frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right)$$

But co-ordinates of the centroid of the  $\Delta ABC$  are  $(\alpha, \beta, \gamma)$  (given).



Therefore,  $\alpha = \frac{a}{3}, \beta = \frac{b}{3}, \gamma = \frac{c}{3}$ , i.e.  $a = 3\alpha, b = 3\beta, c = 3\gamma$

Thus, the equation of plane is

$$\frac{x}{3} + \frac{y}{3} + \frac{z}{3} = 3$$

**Example 10** Find the angle between the lines whose direction cosines are given by the equations:  $3l + m + 5n = 0$  and  $6mn - 2nl + 5lm = 0$ .

**Solution** Eliminating  $m$  from the given two equations, we get

$$\Rightarrow 2n^2 + 3ln + l^2 = 0$$

$$\Rightarrow (n + l)(2n + l) = 0$$

$$\Rightarrow \text{either } n = -l \text{ or } l = -2n$$

Now if  $l = -n$ , then  $m = -2n$

and if  $l = -2n$ , then  $m = n$ .

Thus the direction ratios of two lines are proportional to  $-n, -2n, n$  and  $-2n, n, n$ ,

i.e.  $1, 2, -1$  and  $-2, 1, 1$ .

So, vectors parallel to these lines are

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k} \quad \text{and} \quad \vec{b} = -2\hat{i} + \hat{j} + \hat{k}, \text{ respectively.}$$

If  $\theta$  is the angle between the lines, then

$$\begin{aligned} \cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\ &= \frac{(\hat{i} + 2\hat{j} - \hat{k}) \cdot (-2\hat{i} + \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + (-1)^2} \sqrt{(-2)^2 + 1^2 + 1^2}} = -\frac{1}{6} \end{aligned}$$

Hence  $\theta = \cos^{-1} \left( -\frac{1}{6} \right)$ .

**Example 11** Find the co-ordinates of the foot of perpendicular drawn from the point A (1, 8, 4) to the line joining the points B (0, -1, 3) and C (2, -3, -1).

**Solution** Let L be the foot of perpendicular drawn from the points A (1, 8, 4) to the line passing through B and C as shown in the Fig. 11.2. The equation of line BC by using formula  $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ , the equation of the line BC is

$$\vec{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} - 4\hat{k})$$

$$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = 2\lambda\hat{i} - (2\lambda + 1)\hat{j} + \lambda(3 - 4\lambda)\hat{k}$$

Comparing both sides, we get

$$x = 2\lambda, y = -(2\lambda + 1), z = 3 - 4\lambda \quad (1)$$

Thus, the co-ordinate of L are  $(2\lambda, -(2\lambda + 1), 3 - 4\lambda)$ ,

so that the direction ratios of the line AL are  $(1 - 2\lambda, 8 + (2\lambda + 1), 4 - (3 - 4\lambda))$ , i.e.

$$1 - 2\lambda, 2\lambda + 9, 1 + 4\lambda$$

Since AL is perpendicular to BC, we have,

$$(1 - 2\lambda)(2 - 0) + (2\lambda + 9)(-3 + 1) + (4\lambda + 1)(-1 - 3) = 0$$

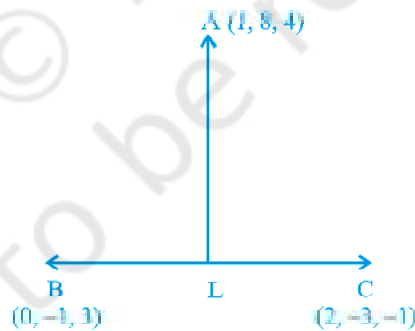


Fig.11.2

$$\Rightarrow \lambda = \frac{-5}{6}$$

The required point is obtained by substituting the value of  $\lambda$ , in (1), which is

$$\left( \frac{-5}{3}, \frac{2}{3}, \frac{19}{3} \right).$$

**Example 12** Find the image of the point (1, 6, 3) in the line  $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ .

**Solution** Let P (1, 6, 3) be the given point and let L be the foot of perpendicular from P to the given line.

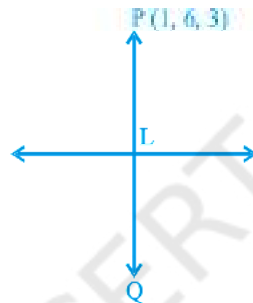


Fig.11.3

The coordinates of a general point on the given line are

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda, \text{ i.e., } x = \lambda, y = 2\lambda + 1, z = 3\lambda + 2.$$

If the coordinates of L are  $(\lambda, 2\lambda + 1, 3\lambda + 2)$ , then the direction ratios of PL are  $\lambda - 1, 2\lambda - 5, 3\lambda - 1$ .

But the direction ratios of given line which is perpendicular to PL are 1, 2, 3. Therefore,  $(\lambda - 1)1 + (2\lambda - 5)2 + (3\lambda - 1)3 = 0$ , which gives  $\lambda = 1$ . Hence coordinates of L are (1, 3, 5).

Let Q  $(x_1, y_1, z_1)$  be the image of P (1, 6, 3) in the given line. Then L is the mid-point of PQ. Therefore,

$$\frac{x_1+1}{2} = 1, \frac{y_1+6}{2} = 3, \frac{z_1+3}{2} = 5$$

$$\Rightarrow x_1 = 1, y_1 = 0, z_1 = 7$$

Hence, the image of (1, 6, 3) in the given line is (1, 0, 7).

**Example 13** Find the image of the point having position vector  $\hat{i} + 3\hat{j} + 4\hat{k}$  in the plane  $\hat{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$ .

**Solution** Let the given point be P  $(\hat{i} + 3\hat{j} + 4\hat{k})$  and Q be the image of P in the plane  $\hat{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$  as shown in the Fig. 11.4.

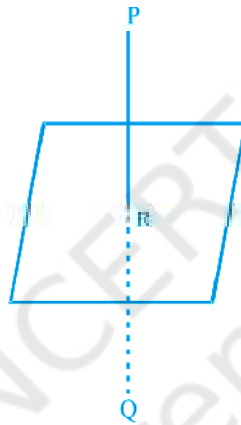


Fig. 11.4

Then PQ is the normal to the plane. Since PQ passes through P and is normal to the given plane, so the equation of PQ is given by

$$\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

Since Q lies on the line PQ, the position vector of Q can be expressed as

$$(\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}), \text{ i.e., } (1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$$

Since R is the mid point of PQ, the position vector of R is

$$\frac{[(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}] + [\hat{i} + 3\hat{j} + 4\hat{k}]}{2}$$

$$\text{i.e., } (\lambda+1)\hat{i} + \left(3 - \frac{\lambda}{2}\right)\hat{j} + \left(4 + \frac{\lambda}{2}\right)\hat{k}$$

Again, since R lies on the plane  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$ , we have

$$\left\{ (\lambda+1)\hat{i} + \left(3 - \frac{\lambda}{2}\right)\hat{j} + \left(4 + \frac{\lambda}{2}\right)\hat{k} \right\} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$$

$$\Rightarrow \lambda = -2$$

Hence, the position vector of Q is  $(\hat{i} + 3\hat{j} + 4\hat{k}) - 2(2\hat{i} - \hat{j} + \hat{k})$ , i.e.  $-3\hat{i} + 5\hat{j} + 2\hat{k}$ .

### Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 14 to 19.

**Example 14** The coordinates of the foot of the perpendicular drawn from the point (2, 5, 7) on the  $x$ -axis are given by

- (A) (2, 0, 0)                      (B) (0, 5, 0)                      (C) (0, 0, 7)                      (D) (0, 5, 7)

**Solution** (A) is the correct answer.

**Example 15** P is a point on the line segment joining the points (3, 2, -1) and (6, 2, -2). If  $x$  co-ordinate of P is 5, then its  $y$  co-ordinate is

- (A) 2                                      (B) 1                                      (C) -1                                      (D) -2

**Solution** (A) is the correct answer. Let P divides the line segment in the ratio of  $\lambda : 1$ ,

$x$  - coordinate of the point P may be expressed as  $x = \frac{6\lambda + 3}{\lambda + 1}$  giving  $\frac{6\lambda + 3}{\lambda + 1} = 5$  so that

$\lambda = 2$ . Thus  $y$ -coordinate of P is  $\frac{2\lambda + 2}{\lambda + 1} = 2$ .

**Example 16** If  $\alpha, \beta, \gamma$  are the angles that a line makes with the positive direction of  $x, y, z$  axis, respectively, then the direction cosines of the line are.

- (A)  $\sin \alpha, \sin \beta, \sin \gamma$                       (B)  $\cos \alpha, \cos \beta, \cos \gamma$   
 (C)  $\tan \alpha, \tan \beta, \tan \gamma$                       (D)  $\cos^2 \alpha, \cos^2 \beta, \cos^2 \gamma$

**Solution** (B) is the correct answer.

**Example 17** The distance of a point P ( $a, b, c$ ) from  $x$ -axis is

- (A)  $\sqrt{a^2 + c^2}$  (B)  $\sqrt{a^2 + b^2}$   
 (C)  $\sqrt{b^2 + c^2}$  (D)  $b^2 + c^2$

**Solution** (C) is the correct answer. The required distance is the distance of P ( $a, b, c$ ) from Q ( $a, 0, 0$ ), which is  $\sqrt{b^2 + c^2}$ .

**Example 18** The equations of  $x$ -axis in space are

- (A)  $x = 0, y = 0$  (B)  $x = 0, z = 0$  (C)  $x = 0$  (D)  $y = 0, z = 0$

**Solution** (D) is the correct answer. On  $x$ -axis the  $y$ - co-ordinate and  $z$ - co-ordinates are zero.

**Example 19** A line makes equal angles with co-ordinate axis. Direction cosines of this line are

- (A)  $\pm (1, 1, 1)$  (B)  $\pm \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$   
 (C)  $\pm \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$  (D)  $\pm \left( \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)$

**Solution** (B) is the correct answer. Let the line makes angle  $\alpha$  with each of the axis. Then, its direction cosines are  $\cos \alpha, \cos \alpha, \cos \alpha$ .

Since  $\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$ . Therefore,  $\cos \alpha = \pm \frac{1}{\sqrt{3}}$

Fill in the blanks in each of the Examples from 20 to 22.

**Example 20** If a line makes angles  $\frac{\pi}{2}, \frac{3}{4}\pi$  and  $\frac{\pi}{4}$  with  $x, y, z$  axis, respectively, then its direction cosines are \_\_\_\_\_

**Solution** The direction cosines are  $\cos \frac{\pi}{2}, \cos \frac{3}{4}\pi, \cos \frac{\pi}{4}$ , i.e.,  $\pm \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ .

**Example 21** If a line makes angles  $\alpha, \beta, \gamma$  with the positive directions of the coordinate axes, then the value of  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$  is \_\_\_\_\_

**Solution** Note that

$$\begin{aligned} \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma &= (1 - \cos^2 \alpha) + (1 - \cos^2 \beta) + (1 - \cos^2 \gamma) \\ &= 3 - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = 2. \end{aligned}$$

**Example 22** If a line makes an angle of  $\frac{\pi}{4}$  with each of  $y$  and  $z$  axis, then the angle which it makes with  $x$ -axis is \_\_\_\_\_

**Solution** Let it makes angle  $\alpha$  with  $x$ -axis. Then  $\cos^2 \alpha + \cos^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} = 1$

which after simplification gives  $\alpha = \frac{\pi}{2}$ .

State whether the following statements are **True** or **False** in Examples 23 and 24.

**Example 23** The points  $(1, 2, 3), (-2, 3, 4)$  and  $(7, 0, 1)$  are collinear.

**Solution** Let A, B, C be the points  $(1, 2, 3), (-2, 3, 4)$  and  $(7, 0, 1)$ , respectively.

Then, the direction ratios of each of the lines AB and BC are proportional to  $-3, 1, 1$ .

Therefore, the statement is true.

**Example 24** The vector equation of the line passing through the points  $(3,5,4)$  and  $(5,8,11)$  is

$$\vec{r} = 3\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 7\hat{k})$$

**Solution** The position vector of the points  $(3,5,4)$  and  $(5,8,11)$  are

$$\vec{a} = 3\hat{i} + 5\hat{j} + 4\hat{k}, \vec{b} = 5\hat{i} + 8\hat{j} + 11\hat{k},$$

and therefore, the required equation of the line is given by

$$\vec{r} = 3\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 7\hat{k})$$

Hence, the statement is true.

### 11.3 EXERCISE

#### Short Answer (S.A.)

- Find the position vector of a point A in space such that  $\overline{OA}$  is inclined at  $60^\circ$  to OX and at  $45^\circ$  to OY and  $|\overline{OA}| = 10$  units.
- Find the vector equation of the line which is parallel to the vector  $3\hat{i} - 2\hat{j} + 6\hat{k}$  and which passes through the point  $(1, -2, 3)$ .
- Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\text{and } \frac{x-4}{5} = \frac{y-1}{2} = z \text{ intersect.}$$

Also, find their point of intersection.

- Find the angle between the lines  
 $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$  and  $\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$
- Prove that the line through A  $(0, -1, -1)$  and B  $(4, 5, 1)$  intersects the line through C  $(3, 9, 4)$  and D  $(-4, 4, 4)$ .
- Prove that the lines  $x = py + q, z = ry + s$  and  $x = p'y + q', z = r'y + s'$  are perpendicular if  $pp' + rr' + 1 = 0$ .
- Find the equation of a plane which bisects perpendicularly the line joining the points A  $(2, 3, 4)$  and B  $(4, 5, 8)$  at right angles.
- Find the equation of a plane which is at a distance  $3\sqrt{3}$  units from origin and the normal to which is equally inclined to coordinate axis.
- If the line drawn from the point  $(-2, -1, -3)$  meets a plane at right angle at the point  $(1, -3, 3)$ , find the equation of the plane.
- Find the equation of the plane through the points  $(2, 1, 0)$ ,  $(3, -2, -2)$  and  $(3, 1, 7)$ .



11. Find the equations of the two lines through the origin which intersect the line

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} \text{ at angles of } \frac{\pi}{3} \text{ each.}$$

12. Find the angle between the lines whose direction cosines are given by the equations  $l + m + n = 0$ ,  $l^2 + m^2 - n^2 = 0$ .

13. If a variable line in two adjacent positions has direction cosines  $l, m, n$  and  $l + \delta l, m + \delta m, n + \delta n$ , show that the small angle  $\delta\theta$  between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

14. O is the origin and A is  $(a, b, c)$ . Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances  $a, b, c$  and  $a', b', c'$ , respectively, from the origin, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}.$$

**Long Answer (L.A.)**

16. Find the foot of perpendicular from the point  $(2, 3, -8)$  to the line

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}. \text{ Also, find the perpendicular distance from the given point to the line.}$$

17. Find the distance of a point  $(2, 4, -1)$  from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$$

18. Find the length and the foot of perpendicular from the point  $\left(1, \frac{3}{2}, 2\right)$  to the plane  $2x - 2y + 4z + 5 = 0$ .

19. Find the equations of the line passing through the point  $(3, 0, 1)$  and parallel to the planes  $x + 2y = 0$  and  $3y - z = 0$ .

20. Find the equation of the plane through the points  $(2,1,-1)$  and  $(-1,3,4)$ , and perpendicular to the plane  $x - 2y + 4z = 10$ .
21. Find the shortest distance between the lines given by  $\vec{r} = (8 + 3\lambda\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k})$  and  $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$ .
22. Find the equation of the plane which is perpendicular to the plane  $5x + 3y + 6z + 8 = 0$  and which contains the line of intersection of the planes  $x + 2y + 3z - 4 = 0$  and  $2x + y - z + 5 = 0$ .
23. The plane  $ax + by = 0$  is rotated about its line of intersection with the plane  $z = 0$  through an angle  $\alpha$ . Prove that the equation of the plane in its new position is  $ax + by \pm (\sqrt{a^2 + b^2} \tan \alpha) z = 0$ .
24. Find the equation of the plane through the intersection of the planes  $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$  and  $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$ , whose perpendicular distance from origin is unity.
25. Show that the points  $(\hat{i} - \hat{j} + 3\hat{k})$  and  $3(\hat{i} + \hat{j} + \hat{k})$  are equidistant from the plane  $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$  and lies on opposite side of it.
26.  $\overline{AB} = 3\hat{i} - \hat{j} + \hat{k}$  and  $\overline{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$  are two vectors. The position vectors of the points A and C are  $6\hat{i} + 7\hat{j} + 4\hat{k}$  and  $-9\hat{j} + 2\hat{k}$ , respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that  $\overline{PQ}$  is perpendicular to  $\overline{AB}$  and  $\overline{CD}$  both.
27. Show that the straight lines whose direction cosines are given by  $2l + 2m - n = 0$  and  $mn + nl + lm = 0$  are at right angles.
28. If  $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$  are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to  $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$  makes equal angles with them.

### Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises from 29 to 36.

29. Distance of the point  $(\alpha, \beta, \gamma)$  from y-axis is

- (A)  $\beta$             (B)  $|\beta|$             (C)  $|\beta| + | |$             (D)  $\sqrt{\alpha^2 + \beta^2}$
- 30.** If the direction cosines of a line are  $k, k, k$ , then
- (A)  $k > 0$             (B)  $0 < k < 1$             (C)  $k = 1$             (D)  $k = \frac{1}{\sqrt{3}}$  or  $-\frac{1}{\sqrt{3}}$
- 31.** The distance of the plane  $\vec{r} \cdot \left( \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k} \right) = 1$  from the origin is
- (A) 1            (B) 7            (C)  $\frac{1}{7}$             (D) None of these
- 32.** The sine of the angle between the straight line  $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$  and the plane  $2x - 2y + z = 5$  is
- (A)  $\frac{10}{6\sqrt{5}}$             (B)  $\frac{4}{5\sqrt{2}}$             (C)  $\frac{2\sqrt{3}}{5}$             (D)  $\frac{\sqrt{2}}{10}$
- 33.** The reflection of the point  $(\alpha, \beta, \gamma)$  in the  $xy$ -plane is
- (A)  $(\alpha, \beta, 0)$             (B)  $(0, 0, \gamma)$             (C)  $(-\alpha, -\beta, \gamma)$             (D)  $(\alpha, \beta, -\gamma)$
- 34.** The area of the quadrilateral ABCD, where A(0, 4, 1), B(2, 3, -1), C(4, 5, 0) and D(2, 6, 2), is equal to
- (A) 9 sq. units            (B) 18 sq. units            (C) 27 sq. units            (D) 81 sq. units
- 35.** The locus represented by  $xy + yz = 0$  is
- (A) A pair of perpendicular lines            (B) A pair of parallel lines  
(C) A pair of parallel planes            (D) A pair of perpendicular planes
- 36.** The plane  $2x - 3y + 6z - 11 = 0$  makes an angle  $\sin^{-1}(\alpha)$  with  $x$ -axis. The value of  $\alpha$  is equal to
- (A)  $\frac{\sqrt{3}}{2}$             (B)  $\frac{\sqrt{2}}{3}$             (C)  $\frac{2}{7}$             (D)  $\frac{3}{7}$

Fill in the blanks in each of the Exercises 37 to 41.

37. A plane passes through the points (2,0,0) (0,3,0) and (0,0,4). The equation of plane is \_\_\_\_\_.
38. The direction cosines of the vector  $(2\hat{i} + 2\hat{j} - \hat{k})$  are \_\_\_\_\_.
39. The vector equation of the line  $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$  is \_\_\_\_\_.
40. The vector equation of the line through the points (3,4,-7) and (1,-1,6) is \_\_\_\_\_.
41. The cartesian equation of the plane  $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$  is \_\_\_\_\_.

State **True** or **False** for the statements in each of the Exercises 42 to 49.

42. The unit vector normal to the plane  $x + 2y + 3z - 6 = 0$  is  $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$ .
43. The intercepts made by the plane  $2x - 3y + 5z + 4 = 0$  on the co-ordinate axis are  $-2, \frac{4}{3}, -\frac{4}{5}$ .
44. The angle between the line  $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$  and the plane  $\vec{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$  is  $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$ .
45. The angle between the planes  $\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 1$  and  $\vec{r} \cdot (\hat{i} - \hat{j}) = 4$  is  $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$ .
46. The line  $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$  lies in the plane  $\vec{r} \cdot (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$ .
47. The vector equation of the line  $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$  is \_\_\_\_\_.

$$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}).$$

- 48.** The equation of a line, which is parallel to  $2\hat{i} + \hat{j} + 3\hat{k}$  and which passes through the point  $(5, -2, 4)$ , is  $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$ .
- 49.** If the foot of perpendicular drawn from the origin to a plane is  $(5, -3, -2)$ , then the equation of plane is  $\vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$ .



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